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Creating a "Dessin Explorer" for Toroidal Belyĭ Pairs

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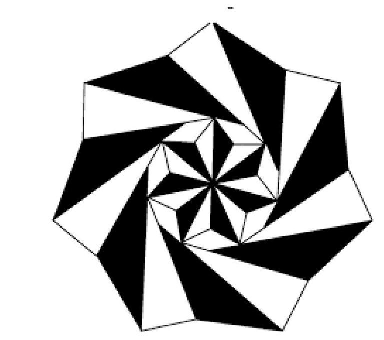
Purdue Research in Mathematics Experience (PRiME)



Abstract

A Belyĭ map $\beta : \mathbb{P}^1(\mathbb{C}) \to \mathbb{P}^1(\mathbb{C})$ is a rational function with at most three critical values; we may assume these values are $\{0,\,1,\,\infty\}$. A Dessin d'Enfant is a planar bipartite graph obtained by considering the preimage of a path between two of these critical values, usually taken to be the line segment from 0 to 1. Such graphs can be drawn on the sphere by composing with stereographic projection: $\beta^{-1}\big([0,1]\big) \subseteq \mathbb{P}^1(\mathbb{C}) \simeq S^2(\mathbb{R})$. Replacing $\mathbb{P}^1$ with an elliptic curve E, there is a similar definition of a Belyĭ map $\beta : E(\mathbb{C}) \to \mathbb{P}^1(\mathbb{C})$. Since $E(\mathbb{C}) \simeq \mathbb{T}^2(\mathbb{R})$ is a torus, we call (E, β) a toroidal Belyĭ pair. The corresponding Dessin d'Enfant can be drawn on the torus by composing with an elliptic logarithm: $\beta^{-1}\big([0,1]\big) \subseteq E(\mathbb{C}) \simeq \mathbb{T}^2(\mathbb{R})$.

This project seeks to create software which will compute (i) Belyĭ pairs (X, β) for either $X = \mathbb{P}^1(\mathbb{C}) \simeq S^2(\mathbb{R})$ or $X = E(\mathbb{C}) \simeq \mathbb{T}^2(\mathbb{R})$, (ii) their corresponding Dessins d'Enfant, and (iii) their monodromy groups. There is preliminary software which partially does this in `Mathematica`; this project seeks to port and expand the code in `Sage`. This software would allow individuals to explore the properties of Belyĭ maps and their Dessins d'Enfants.

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Belyĭ Maps

A **Belyĭ map** $\beta : X \to \mathbb{P}^1(\mathbb{C})$ is a morphism from a compact, connected Riemann surface X which is unramified away from $\{0,\,1,\,\infty\}$. Using the Riemann-Roch Theorem, we can and always do assume the Riemann surface X is a projective variety. This means there are homogeneous polynomials f, p, q such that $X :\ f(x,y) = 0$ and $\beta = p/q$. In particular, β must be a non-constant rational function, so the sets $B = \beta^{-1}(0)$, $W = \beta^{-1}(1)$, and $F = \beta^{-1}(\infty)$ are each finite.

Dessin d'Enfants

A **Dessin d'Enfant** Δ is a bipartite graph of genus g which can be embedded on a compact, connected Riemann suface X without crossings. Denoting B as the collection of "black" vertices, W as the collection of "white" vertices, and F as the collection of (midpoints of) faces, the Euler characteristic asserts that $N = |B| + |W| + |F| + (2\,g - 2)$ is the number of edges of such a graph.

Monodromy Groups

A **Monodromy Group** is a triple $(\sigma_0,\,\sigma_1,\,\sigma_\infty)$ of permutations in a symmetric group S_N on N letters which satisfies $\sigma_0 \circ \sigma_1 \circ \sigma_\infty = 1$. In particular, the group $G = \langle \sigma_0,\,\sigma_1,\,\sigma_\infty \rangle$ generated by them is a subgroup of S_N.

Degree Sequences

A multiset of three multisets of positive integers

$$\mathcal{D} = \Big\{ \big\{ e_P \,\big|\, P \in B \big\},\ \big\{ e_P \,\big|\, P \in W \big\},\ \big\{ e_P \,\big|\, P \in F \big\} \Big\}$$

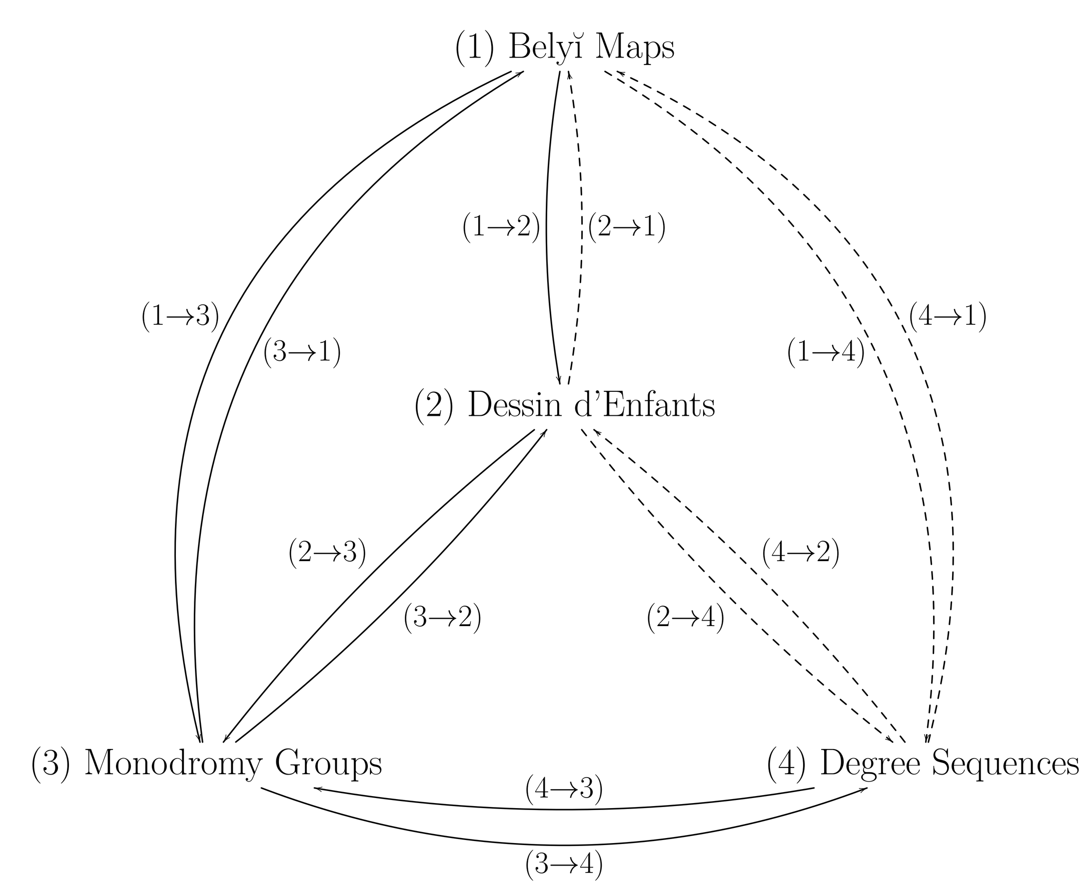
is said to be a **Degree Sequence** if there are nonnegative integers N and g such that

$$N = \sum_{P \in B} e_P = \sum_{P \in W} e_P = \sum_{P \in F} e_P = |B| + |W| + |F| + (2\,g - 2).$$

It follows from the Riemann-Hurwitz Genus formula that this relation is a necessary condition if $\mathcal{D}$ is to be associated to a Belyĭ map $\beta : X \to \mathbb{P}^1(\mathbb{C})$ for a compact, connected Riemann surface X of genus g. In particular, $\mathcal{D}$ is a multiset of three partitions of N.

Belyĭ Map / Dessin d'Enfant / Monodromy Explorer

For each of the four objects above, find effective algorithms to compute all other three. That is, find effective algorithms for the following 12 arrows.



There is preliminary software which partially does this in `Mathematica`, although we wish to port this to `Sage`.

(1) From Belyĭ Maps ...

(2) **. . . To Dessin d'Enfants.** Choose a small $\varepsilon > 0$, and consider the finite set

$$\Delta = \bigcup_{a=0}^{b} \left\{ (x:y:1) \in \mathbb{P}^2(\mathbb{C}) \ \middle|\ f(x,y) = b\,p(x,y) - a\,q(x,y) = 0 \right\} \approx \beta^{-1}\big([0,1]\big)$$

in terms of the positive integer $b = \lfloor 1/\varepsilon \rfloor$. Then $\Delta \hookrightarrow X$ is the Dessin d'Enfant for β.

(3) **. . . To Monodromy Groups.** Fix $y_0 \in \mathbb{P}^1(\mathbb{C})$ different from 0, 1, ∞; and define $\beta^{-1}(y_0) = \{P_1,\,P_2,\,\dots,\,P_N\}$. We construct $2\,N$ functions via the differential equations

$$\left\{ \begin{aligned} \frac{d\vec{\gamma}_0^{(i)}}{dt} &= \frac{2\,\pi\,\sqrt{-1}\,p\,q}{q\left(\dfrac{\partial f}{\partial x}\dfrac{\partial p}{\partial y} - \dfrac{\partial f}{\partial y}\dfrac{\partial p}{\partial x}\right) - p\left(\dfrac{\partial f}{\partial x}\dfrac{\partial q}{\partial y} - \dfrac{\partial f}{\partial y}\dfrac{\partial q}{\partial x}\right)} \begin{bmatrix} -\dfrac{\partial f}{\partial y} \\ +\dfrac{\partial f}{\partial x} \end{bmatrix} \\ \vec{\gamma}_0^{(i)}(0) &= P_i \end{aligned} \right.$$

$$\left\{ \begin{aligned} \frac{d\vec{\gamma}_1^{(i)}}{dt} &= \frac{2\,\pi\,\sqrt{-1}\,(p-q)\,q}{q\left(\dfrac{\partial f}{\partial x}\dfrac{\partial p}{\partial y} - \dfrac{\partial f}{\partial y}\dfrac{\partial p}{\partial x}\right) - p\left(\dfrac{\partial f}{\partial x}\dfrac{\partial q}{\partial y} - \dfrac{\partial f}{\partial y}\dfrac{\partial q}{\partial x}\right)} \begin{bmatrix} -\dfrac{\partial f}{\partial y} \\ +\dfrac{\partial f}{\partial x} \end{bmatrix} \\ \vec{\gamma}_1^{(i)}(0) &= P_i \end{aligned} \right.$$

Each system has a unique solution. Now compute the triple $(\sigma_0,\,\sigma_1,\,\sigma_\infty)$ in terms of the permutations $\sigma_0,\,\sigma_1,\,\sigma_\infty \in S_N$ which satisfy the relations

$$\vec{\gamma}_0^{(i)}(1) = P_{\sigma_0(i)}, \qquad \vec{\gamma}_1^{(i)}(1) = P_{\sigma_1(i)}, \qquad \text{and} \qquad \sigma_\infty = \sigma_1^{-1} \circ \sigma_0^{-1}.$$

(4) **. . . To Degree Sequences.** Once we have the monodromy group $(\sigma_0,\,\sigma_1,\,\sigma_\infty)$, we can compute the Degree sequence $\mathcal{D}$ as in $(3 \to 4)$.

(2) From Dessin d'Enfants ...

(1) **. . . To Belyĭ Maps.** Starting with a Dessin d'Enfant, we compute its monodromy group as in $(2 \to 3)$. John Voight and others [8], [11] have code which computes Belyĭ maps from a given monodromy in $(3 \to 1)$.

(3) **. . . To Monodromy Groups.** Label the edges from 1 through N. Since the compact, connected surface X is oriented, read off the labels counter-clockwise of the edges incident to each vertex $P \in B$ ($P \in W$, respectively) to find the integers $B_{P,1}$, $B_{P,2}$, $\dots$, B_{P,e_P} ($W_{P,1}$, $W_{P,2}$, $\dots$, W_{P,e_P}, respectively). Then the permutations

$$\sigma_0 = \prod_{P \in B} \big(B_{P,1}\ B_{P,2}\ \cdots\ B_{P,e_P}\big)$$

$$\sigma_1 = \prod_{P \in W} \big(W_{P,1}\ W_{P,2}\ \cdots\ W_{P,e_P}\big)$$

$$\sigma_\infty = \sigma_1^{-1} \circ \sigma_0^{-1}$$

form the desired triple $(\sigma_0,\,\sigma_1,\,\sigma_\infty)$ which satisfies $\sigma_0 \circ \sigma_1 \circ \sigma_\infty = 1$. Mark van Hoeij [5], [6] has code which does this very quickly.

(4) **. . . To Degree Sequences.** Once we have the monodromy group $(\sigma_0,\,\sigma_1,\,\sigma_\infty)$, we compute the Degree sequence $\mathcal{D}$ as in $(3 \to 4)$.

(3) From Monodromy Groups ...

(1) **. . . To Belyĭ Maps.** John Voight and his graduate students [8], [11] have this implemented this step.

(2) **. . . To Dessin d'Enfants.** Express the three given permutations as a product of disjoint cycles:

$$\sigma_0 = \prod_{P \in B} \big(B_{P,1}\ B_{P,2}\ \cdots\ B_{P,e_P}\big)$$

$$\sigma_1 = \prod_{P \in W} \big(W_{P,1}\ W_{P,2}\ \cdots\ W_{P,e_P}\big)$$

$$\sigma_\infty = \prod_{P \in F} \big(F_{P,1}\ F_{P,2}\ \cdots\ F_{P,e_P}\big)$$

Place $|B|$ vertices P on X and color them "black", then draw e_P edges adjacent to each $P \in B$. Going counter-clockwise, label these edges the integers $B_{P,1}$, $B_{P,2}$, $\dots$, B_{P,e_P}. Similarly, place $|W|$ vertices P on X and color them "white", then draw e_P edges adjacent to each $P \in W$. Going counter-clockwise, label these edges the integers $W_{P,1}$, $W_{P,2}$, $\dots$, W_{P,e_P}. Connect the edges with the same integer label, then move the vertices $P \in B \cup W$ as necessary so that there are $|F|$ faces. This is implemented in `Sage`.

(4) **. . . To Degree Sequences.** Express the three given permutations as a product of disjoint cycles as above. The desired degree sequence is that multiset formed by the lengths of the cycles, that is,

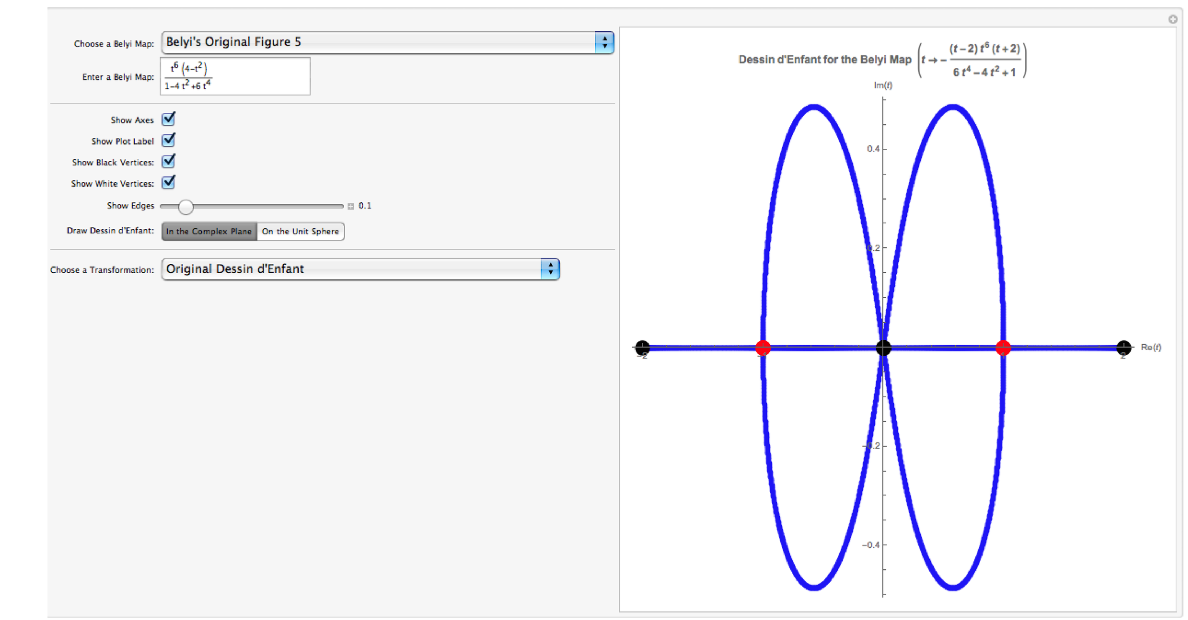
$$\mathcal{D} = \Big\{ \big\{ e_P \,\big|\, P \in B \big\},\ \big\{ e_P \,\big|\, P \in W \big\},\ \big\{ e_P \,\big|\, P \in F \big\} \Big\}.$$

(4) From Degree Sequences ...

(1) **. . . To Belyĭ Maps.** Compute the monodromy group as in $(4 \to 3)$. Then compute the Belyĭ map as in $(3 \to 1)$.

(2) **. . . To Dessin d'Enfants.** Once we have the monodromy group as in $(4 \to 3)$, then we can compute the Dessin d'Enfant as in $(3 \to 2)$.

(3) **. . . To Monodromy Groups.** Search through all triples $(\sigma_0,\,\sigma_1,\,\sigma_\infty)$ of permutations in a symmetric group S_N which are the product of disjoint cycles as above and which satisfies $\sigma_0 \circ \sigma_1 \circ \sigma_\infty = 1$.



Dessin Explorer

`http://www.math.purdue.edu/~egoins/notes/dessin_explorer.cdf`

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